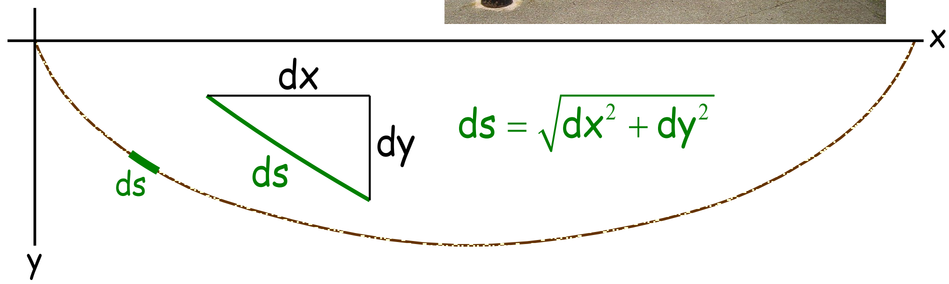
Derivation of the Catenary Curve
Using the Calculus of Variations

Use the Calculus of Variations to find the catenary curve of a weighted chain.

Write out the potential energy and minimize it



$$dU = gydm = gy(\lambda ds) = g\lambda yds$$

$$U = g\lambda \int yds$$
$$= g\lambda \int y\sqrt{dx^2 + dy^2}$$
$$= g\lambda \int y\sqrt{1+(y')^2}\,dx$$

The functional is $f = y\sqrt{1+(y')^2}$ so take derivatives

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(y\sqrt{1+(y')^2}\right) = \sqrt{1+(y')^2}$$

$$\frac{\partial f}{\partial y'} = \frac{\partial}{\partial y'}\left(y\sqrt{1+(y')^2}\right) = \frac{y(2y')}{2\sqrt{1+(y')^2}} = \frac{yy'}{\sqrt{1+(y')^2}}$$

Substitute these into Euler's Equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0$$

$$\sqrt{1+(y')^2} - \frac{d}{dx}\left(\frac{yy'}{\sqrt{1+(y')^2}}\right) = 0$$

$$\sqrt{1+(y')^2} = \left[\frac{(y')^2 + yy''}{\sqrt{1+(y')^2}}\right] + \left\{yy'\left[\frac{-1}{2}\right]\left[1+(y')^2\right]^{-3/2} 2(y')y''\right\}$$

$$\sqrt{1+(y')^2} = \left[\frac{(y')^2 + yy''}{\sqrt{1+(y')^2}}\right] - \left\{\frac{y(y')^2 y''}{\left[1+(y')^2\right]^{-3/2}}\right\}$$

$$\left(1+(y')^2\right)^2 = \left[(y')^2 + yy''\right]\left[1+(y')^2\right] - y(y')^2 y''$$

$$1 + 2(y')^2 + (y')^4 = (y')^2 + yy'' + \left[(y')^2 + yy''\right](y')^2 - y(y')^2 y''$$

$$1 + 2(y')^2 + (y')^4 = (y')^2 + yy'' + (y')^4 + yy''(y')^2 - y(y')^2 y''$$

$$1 + (y')^2 - yy'' = 0$$

Rearrage this to something we can solve:

$$1+(y')^2 - yy'' = 0$$
$$\left[1+(y')^2 = yy''\right]y'$$
$$y'+(y')^3 = yy'y''$$
$$\left(1+(y')^2\right)y' = yy'y''$$
$$\frac{y'}{y} = \frac{y'y''}{\left(1+(y')^2\right)}$$
$$\frac{y'}{y} - \frac{y'y''}{1+(y')^2} = 0$$

Each term is actually a derivative

$$\frac{y'}{y} = \frac{d}{dx}\ln(y) = \left[\ln(y)\right]'$$
$$\frac{y'y''}{1+(y')^2} = \frac{1}{2}\frac{d}{dx}\ln\left(y'^2+1\right) = \left[\frac{1}{2}\ln\left(y'^2+1\right)\right]'$$

So our DE is

$$\left[\ln(y)\right]' = \left[\frac{1}{2}\ln\left(y'^2+1\right)\right]'$$

Which gives

$$\ln(y) + A = \frac{1}{2}\ln\left(y'^2+1\right)$$
$$2\ln(y) + 2A = \ln\left(y'^2+1\right)$$
$$\ln\left(y^2\right) + \ln(C) = \ln\left(y'^2+1\right)$$
$$\ln\left(Cy^2\right) = \ln\left(y'^2+1\right)$$
$$Cy^2 = y'^2 + 1$$

Rearranging to get a separable equation,

$$Cy^2 - 1 = y'^2$$
$$\frac{y'}{\sqrt{Cy^2-1}} = 1$$
$$\frac{dy}{\sqrt{Cy^2-1}} = dx$$

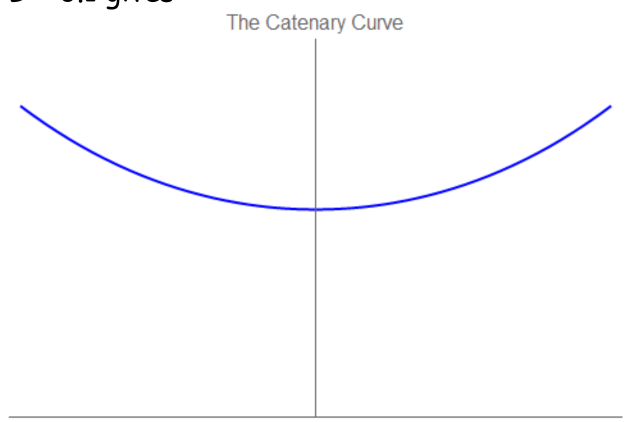
The CRC handbook derivative #32 (p. 184)

$$\frac{d}{dx}\cosh^{-1}u = \frac{1}{\sqrt{u^2-1}}\frac{du}{dx}$$

So, at last we see that

$$\frac{1}{\sqrt{C}}\cosh^{-1}\left(\sqrt{C}\,y\right) = x + A$$
$$y = A\cosh(Cx) + B$$

Where A, B and C are arbitrary constants. Plotting this in Mathematica with A=C=1 and B = 0.1 gives:



Whew!